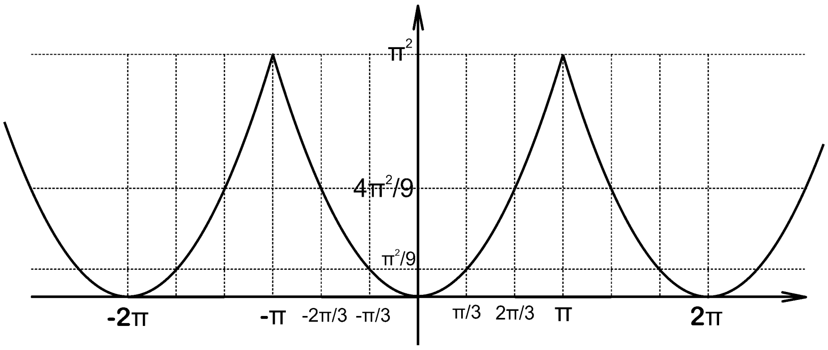
Analize III, pismeni ispit, 07.07.2014.



1. Funkciju definisanu grafikom razviti u Furijer-ov red. Dobijeni rezultat iskoristiti za sumiranje redova

(a) $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + ...;$

(b) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + ...$

2. Izračunati $\iint\limits_D y\,\mathrm{d}x\mathrm{d}y$ gdje je $D = \{(x,y) : x^2+y^2 \le 1,\ x^2+y^2 \le 2x,\ y \ge 0\}$.

3. Isključivo primjenom krivoliniskog integrala prve vrste izračunati površinu djela cilindra $x^2 + y^2 = Rx$ koji se nalazi unutar sfere $x^2+y^2+z^2 = R^2$.

4. Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v} = 2x(y^2+z^2)\vec{i} + 2y(x^2+z^2)\vec{j} + 2z(x^2+y^2)\vec{k}.$$

VAŽNO: Ovaj papir treba predati zajedno s rješenjima zadataka! Prije rješenja prepisati postavku (tekst) zadatka. Ispit pisati isključivo hemijskom olovkom plave ili crne tinte.

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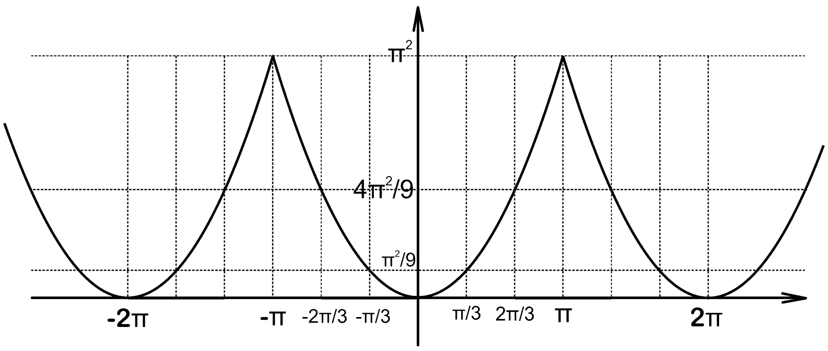

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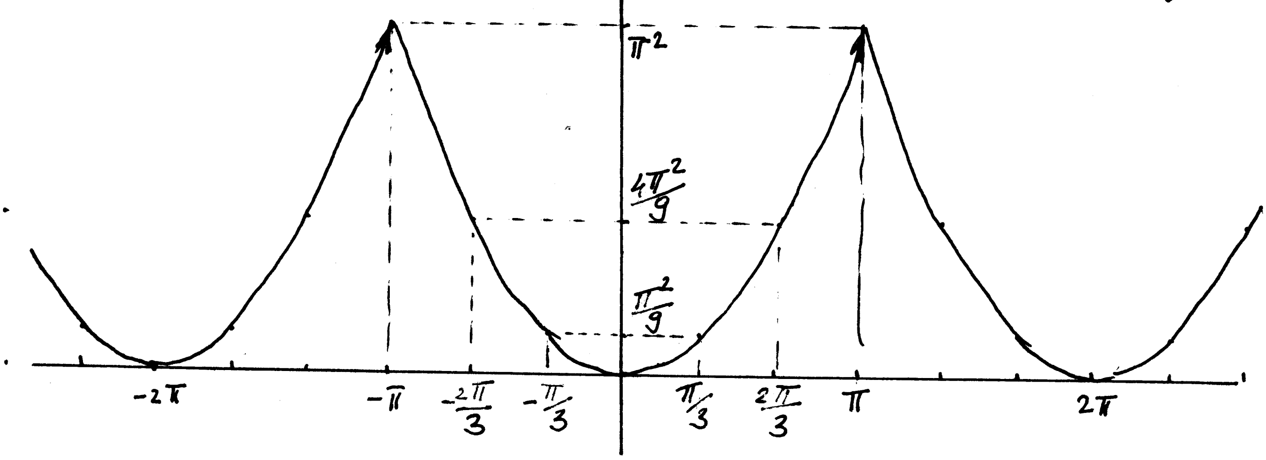
(#) F-ju definisanu grafikom razviti u Furijer-ov red.



Dobijeni rezultat iskoristiti za sumiranje redova

a) $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots$; b) $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots$.

Rj. Primjetimo da je f-ja periodična perioda 2π, pa je možemo razviti u Furijeov red.

Posmatrajmo f-ju na intervalu $(-\pi,\pi)$.

$f(-\pi)=\pi^2$ $\quad f(0)=0$

$f(-\frac{2\pi}{3})=\frac{4\pi^2}{9}$ $\quad f(\frac{\pi}{3})=\frac{\pi^2}{9}$

$f(-\frac{\pi}{3})=\frac{\pi^2}{9}$

Primjetimo da je $f(x)=x^2$, za $x\in(-\pi,\pi)$.

Dovoljno ju je pretvoriti u Furijer-ov red na ovom intervalu.

Prisjetimo se: Trigonometrijski red oblika

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{b-a}+b_n\sin\frac{2n\pi x}{b-a}\right),\quad x\in[a,b]$$

zovemo Furijer-ov red f-je $f(x)$ na intervalu $[a,b]$, gdje su $a_0=\frac{2}{b-a}\int_a^b f(x)\,dx$, $a_n=\frac{2}{b-a}\int_a^b f(x)\cos\frac{2n\pi x}{b-a}\,dx$

i $b_n=\frac{2}{b-a}\int_a^b f(x)\sin\frac{2n\pi x}{b-a}\,dx$ brojevi koje zovem Furijer-ovi koeficijenti.

Posmatramo interval $[-\pi, \pi]$, $b-a=2\pi$, $\frac{2}{b-a}=\frac{1}{\pi}$, $\frac{2n\pi x}{b-a}=nx$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx=\frac{1}{\pi}\cdot\frac{1}{3}x^3\Big|_{-\pi}^{\pi}=\frac{2\pi^3}{3\pi}=\frac{2}{3}\pi^2$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{parna}}{x^2}\,\underset{\text{parna}}{\cos nx}}_{\text{parna}}\,dx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx=\begin{vmatrix} u=x^2 & dv=\cos nx\,dx \\ du=2x & v=\frac{1}{n}\sin nx \end{vmatrix}$$

$$=\frac{2}{\pi}\left[\underbrace{\frac{1}{n}x^2\sin nx\Big|_0^{\pi}}_{=0-0}-\frac{2}{n}\int_0^{\pi}x\sin nx\,dx\right]=\begin{vmatrix} u=x & dv=\sin nx\,dx \\ du=dx & v=\frac{-1}{n}\cos nx \end{vmatrix}=$$

$$=-\frac{4}{n\pi}\left[-\frac{1}{n}x\cos nx\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx\right]=\frac{4}{n^2\pi}(\pi\cos n\pi-0)+\underbrace{\frac{1}{n^2}\sin nx\Big|_0^{\pi}}_{=0-0}$$

$$=(-1)^n\frac{4}{n^2},\quad n\neq 0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\underset{\text{parna}}{x^2}\,\underset{\text{neparna}}{\sin nx}}_{=\text{neparna}}\,dx=0$$

traženi Furijer-ov red

Prema tome $$x^2\sim\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}$$

Za $x=0$ imamo

$$0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\frac{\pi^2}{3}+4\left(\frac{-1}{1}+\frac{1}{2^2}-\frac{1}{3^2}+\dots\right)$$

$$\Rightarrow 1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\dots=\frac{\pi^2}{12}$$

Za $x=\pi$ imamo

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\dots+\frac{1}{n^2}+\dots=\frac{\pi^2}{6}$$

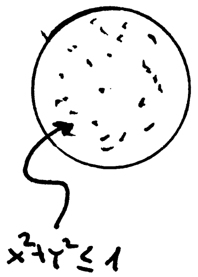
Izračunati $\iint\limits_D y\,dx\,dy$ gdje je $D = \{(x;y) : x^2+y^2 \le 1,\ x^2+y^2 \le 2x,\ y \ge 0\}$

Rj:

$D = \{(x;y) : x^2+y^2 \le 1,\ x^2+y^2 \le 2x,\ y \ge 0\}$

$x^2+y^2 \le 1 \Downarrow 1 = x^2+y^2$

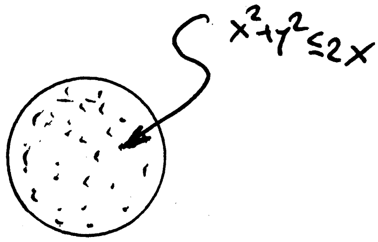
krug sa centrom u $C(0;0)$ polupr. $r=1$



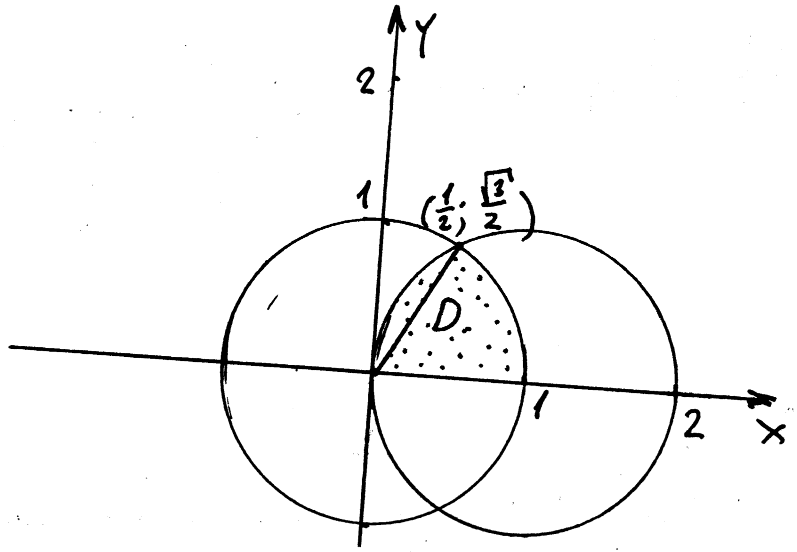
$x^2+y^2 \le 2x \Downarrow x^2+y^2 = 2x$

$x^2 - 2\cdot x + 1 + y^2 = 1$

$(x-1)^2 + y^2 = 1$ krug sa centrom u $C(1;0)$ polupr. $r=1$



Skicirajmo oblast D



Ako uvedemo polarne koordinate

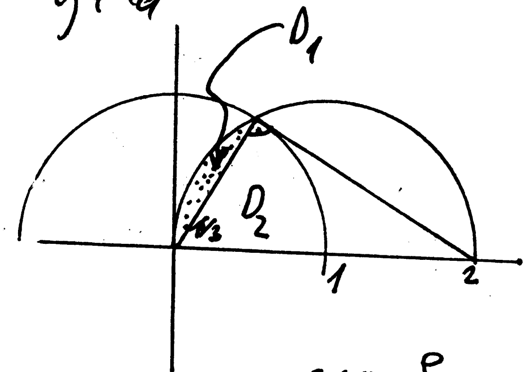
$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$dx\,dy = \rho\,d\rho\,d\varphi$

$D \xrightarrow{\text{transf.}} D'$

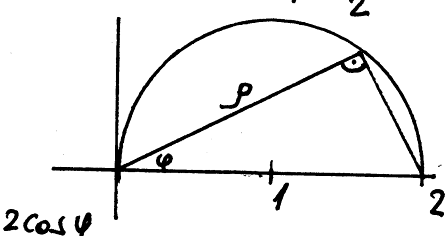
oblast D' možemo podjeliti na dva dijela



$D_2 : \begin{cases} 0 \le \rho \le 1 \\ 0 \le \varphi \le \pi/3 \end{cases}$

$D_1 : \begin{cases} \frac{\pi}{3} \le \varphi \le \frac{\pi}{2} \\ 0 \le \rho \le 2\cos\varphi \end{cases}$

$\cos\varphi = \frac{\rho}{2}$



$\rho = 2\cos\varphi$

$$\iint\limits_{D} y\,dx\,dy = \left| \begin{matrix} \text{uvodimo} \\ \text{polarne} \\ \text{koordinate} \end{matrix} \right| = \iint\limits_{D'} \rho\sin\varphi\,\rho\,d\rho\,d\varphi =$$

$$= \iint\limits_{D_1\cup D_2} \rho^2\sin\varphi\,d\rho\,d\varphi = \iint\limits_{D_1} \rho^2\sin\varphi\,d\rho\,d\varphi + \iint\limits_{D_2} \rho^2\sin\varphi\,d\rho\,d\varphi$$

$$\iint\limits_{D_1} \rho^2\sin\varphi\,d\rho\,d\varphi = \int_{\pi/3}^{\pi/2} \sin\varphi\,d\varphi \int_0^{2\cos\varphi} \rho^2\,d\rho = \frac{1}{3}\int_{\pi/3}^{\pi/2} \sin\varphi\,\rho^3\Big|_0^{2\cos\varphi} d\varphi$$

$$= \frac{8}{3}\int_{\pi/3}^{\pi/2} \cos^3\varphi\sin\varphi\,d\varphi = -\frac{8}{3}\int_{\pi/3}^{\pi/2} \cos^3\varphi\,d(\cos\varphi) = -\frac{8}{3}\cdot\frac{1}{4}\cos^4\varphi\Big|_{\pi/3}^{\pi/2} = -\frac{2}{3}\left(0-\left(\frac{1}{2}\right)^4\right)$$

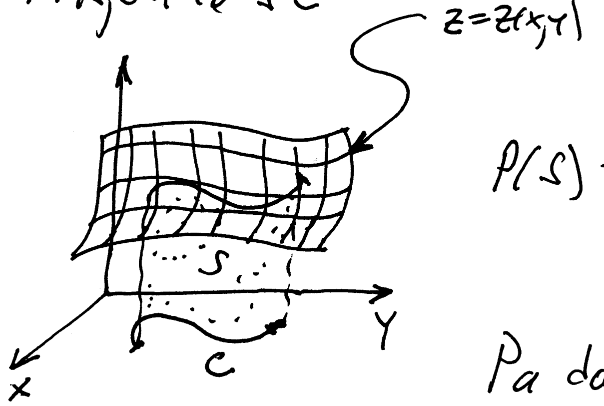
$$= \frac{2}{3}\cdot\frac{1}{16} = \frac{1}{24}$$

$$\iint\limits_{D_2} \rho^2\sin\varphi\,d\rho\,d\varphi = \int_0^{\pi/3} \sin\varphi \int_0^1 \rho^2\,d\rho = \frac{1}{3}\rho^3\Big|_0^1 \cdot \underbrace{(-\cos\varphi)\Big|_0^{\frac{\pi}{3}}}_{-\left(\frac{1}{2}-1\right)} = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

$$\iint\limits_{D} y\,dx\,dy = \frac{1}{24} + \frac{1\cdot 4}{6\cdot 4} = \frac{5}{24} \quad \text{traženo rješenje}$$

(#) Isključivo primjenom krivoliniskog integrala prve vrste izračunati površinu djela cilindra $x^2+y^2=Rx$ koji se nalazi unutar sfere $x^2+y^2+z^2=R^2$.

Rj: Prisjetimo se



$$P(S)=\int_C z(x,y)\,ds$$

Pa da bi odredili površinu cilindra koji se nalazi unutar sfere potrebno je prvo napraviti ortogonalnu projekciju krive $\begin{cases} x^2+y^2=Rx \\ x^2+y^2+z^2=R^2 \end{cases}$ na xOy ravan.

Iz datih tijela (ili iz presjeka datih tijela sa koordinatnim ravnima) nije teško vidjeti da je ortogonalna projekcija krug $x^2+y^2=Rx$,

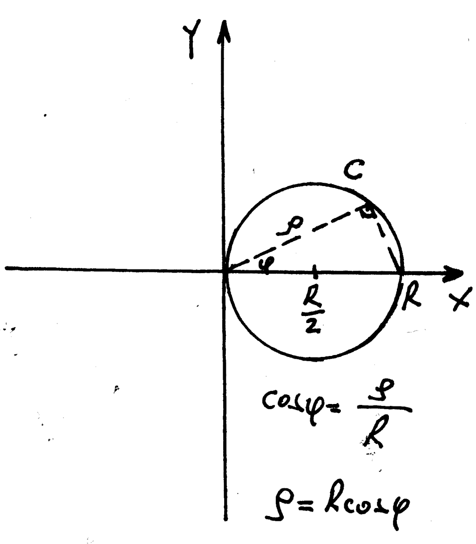
$x^2+y^2=Rx$

$x^2-2\cdot x\cdot\frac{1}{2}R+\left(\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2$

$\left(x-\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2$

Prema tome imamo

$$P=\int_{\left(x-\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2} \pm\sqrt{R^2-x^2-y^2}\,ds = \left|\begin{array}{l} \text{parametrizujmo datu krivu} \\ x=R\cos^2\varphi \\ y=R\sin\varphi\cos\varphi \\ 0\le\varphi\le 2\pi \end{array}\right.$$



$\cos\varphi=\frac{\rho}{R}$

$\rho=R\cos\varphi$

$$\left|\begin{array}{l} ds=\sqrt{(2R\cos\varphi(-\sin\varphi))^2+(R\cos2\varphi)^2}=R \\ x^2+y^2=R^2\cos^4\varphi+R^2\sin^2\varphi\cos^2\varphi=R^2\cos^2\varphi \\ R^2-x^2-y^2=R^2(1-\cos^2\varphi)=R^2\sin^2\varphi \end{array}\right| =$$

$$=4R\cdot R\int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi=4R^2 \text{ tražena površina}$$

(#) Dokazati da je vektorsko polje potencijalno i naći njegov potencijal:

$$\vec{v} = 2x(y^2+z^2)\vec{i} + 2y(x^2+z^2)\vec{j} + 2z(x^2+y^2)\vec{k}.$$

Rj. Vektorsko polje $\vec{v}$ je potencijalno ako je $rot\vec{v} = \vec{0}$.
Rotor vektorskog polja $rot\vec{v}$ se računa . . .

$$rot\vec{v} = \nabla \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

(vektorski proizvod Nabla (∇) operatora i vektorskog polja $\vec{v}$)

$V_x = 2x(y^2+z^2)$ | $V_y = 2y(x^2+z^2)$ | $V_z = 2z(x^2+y^2)$

$\frac{\partial V_x}{\partial y} = 4xy$ | $\frac{\partial V_y}{\partial x} = 4xy$ | $\frac{\partial V_z}{\partial x} = 4xz$

$\frac{\partial V_x}{\partial z} = 4xz$ | $\frac{\partial V_y}{\partial z} = 4yz$ | $\frac{\partial V_z}{\partial y} = 4yz$

$$rot\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix} = \vec{i}(4yz-4yz) - \vec{j}(4xz-4xz) + \vec{k}(4xy-4xy)$$

$= (0,0,0) = \vec{0}$ vektorsko polje je potencijalno

Potencijal polja $\vec{v}$ je f-ja u za koju vrijedi $\vec{v} = grad\,u$.

$$grad\,u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= 2x(y^2+z^2) \\ \frac{\partial u}{\partial y} &= 2y(x^2+z^2) \\ \frac{\partial u}{\partial z} &= 2z(x^2+y^2) \end{aligned}\right\} \dots (1)$$

$u = u(x,y,z)$

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$u = \int 2x(y^2+z^2)\,dx + \varphi(y,z)$

$u = x^2(y^2+z^2) + \varphi(y,z)$

$$\left.\begin{aligned} \frac{\partial u}{\partial y} &= 2yx^2 + \varphi'_y \\ \frac{\partial u}{\partial z} &= 2x^2z + \varphi'_z \end{aligned}\right\} \dots (2)$$

(1) i (2) $\Rightarrow$ $\varphi'_y = 2yz^2$, $\varphi'_z = 2zy^2$ ... (*)

Odredimo f-ju φ: $\varphi = \int 2yz^2\,dy + \psi(z)$

$\varphi = y^2z^2 + \psi(z)$

$\varphi' = 2y^2z + \psi'$ ... (**)

(*) i (**) $\Rightarrow$ $\psi'_z = 0 \Rightarrow \psi(z) = C$

$\Rightarrow \varphi = y^2z^2 + C \Rightarrow u = x^2y^2 + x^2z^2 + y^2z^2 + C$

Potencijal vektorskog polja je $u = x^2y^2 + x^2z^2 + y^2z^2 + C$